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On Berenstein-Douglas-Seiberg duality

Volker Braun
Ecole Normale Supérieure
24 rue Lhomond, 75231 Paris, France
E-mail: volker.braun@lpt.ens.fr

Abstract: I review the proposal of Berenstein-Douglas for a completely general definition of Seiberg duality. To give evidence for their conjecture I present the first example of a physical dual pair and explicitly check that it satisfies the requirements. Then I explicitly show that a pair of toric dual quivers is also dual according to their proposal. All these computations go beyond tilting modules, and really work in the derived category. I introduce all necessary mathematics where needed.

Keywords: Duality in Gauge Field Theories, D-branes, String Duality.

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Contents

1. Introduction

There exists a wealth of knowledge about $\mathcal{N}=1$ field theories (see e.g. [1]), none of which can be proven in any rigorous sense. Of course the handwaving nature of the arguments makes it hard to verify or falsify anything. Yet recently Berenstein and Douglas [2] proposed an exact criterion to decide whether two theories can be (generalized) Seiberg dual. Their proposal is elegant to formulate yet not so obvious to check in practice. The purpose of this paper is to study the implications for physical quivers, as opposed to the toy model in [2].

Since there is no useful introduction I review their proposal and explain the ideas involved. I also revisit the example that they give to provide evidence for their conjecture.

While computationally simple, their example suffers from a serious illness: It requires F-term constraints which do not come from a superpotential, so if at all it can only be a subsector of the full quiver. To remedy this I exhibit a sample pair of quivers which are completely physical.

I then set out to explicitly check that my example is (generalized) Seiberg dual according to the proposal. It turns out that one really has to work in the derived category, i.e. use tilting complexes instead of just tilting modules. Because this is certainly not standard knowledge amongst physicist I carefully explain how to do this in an attempt to make everything self-contained (for another very nice introduction see [3]).

After a lightning review of toric duality I then explicitly check that one pair of toric dual quivers is BDS-dual. This works in the same way as in the previous example, but is technically more challenging.

Finally I discuss some necessary conditions for two quivers to be dual which can sometimes be understood from physical intuition. I prove that one can not simply "fix" the example of [2] by adding an additional arrow.

2. Review of the BDS-duality conjecture

2.1 Beyond representations

The duality conjecture is based on properly distinguishing between the "theory" and its representation. Of course physicists in general ignore such subtleties, not completely without reason (e.g. group theory vs. representation theory of groups). However in this case it is important to make this distinction.

As is well-known, the $\mathcal{N} = 1$ SYM theory for n gauge groups and k bifundamental chiral multiplets is defined by the lagrangian

$$\mathcal{L} = \operatorname{Im}\operatorname{tr}\left[\sum_{i=1}^{n}\int \mathrm{d}^2\theta\, W^{(i)}_{\alpha} W^{(i),\alpha} + \sum_{j=1}^{k}\int \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, \Phi^{(j),\dagger} e^{V^{(j\text{-fund})}} \Phi^{(j)} e^{-V^{(j\text{-anti-fund})}}\right] + \\ + \operatorname{Re}\int \mathrm{d}^2\theta\, W(\Phi_1,\ldots,\Phi_k) \tag{2.1}$$

Given the lagrangian one can expand it in component fields, write down F- and D-term constraints, derive Feynman rules etc. However there is one piece of information missing if one wants to compute actual numbers instead of algebraic manipulations: Nothing in (2.1) tells you the ranks of the gauge groups[1] $\mathrm{U}(N_i)$ and the vevs of the bifundamental fields. This choice of dimensions for vector spaces and explicit matrices is the representation data that we need to completely specify the physics.

This distinction is exactly mirrored by the theory of quivers and their representations. Recall that you can write (2.1) in the following graphical notation:

1. For each factor of the gauge group draw a node.

[1] I will restrict myself to unitary gauge groups throughout this paper

<table>
<tr><th></th><th>Algebraic data</th><th>Representation data</th></tr>
<tr><td>$\mathcal{N}=1$
SYM</td><td>$$\begin{aligned} \mathcal{L} &= \operatorname{Im}\operatorname{tr}\sum_{i=1}^{3}\int \mathrm{d}^2\theta\,\big(W^{(i)}\big)^2 + \\ &+\cdots+\operatorname{Re}\int \mathrm{d}^2\theta\,W \\ W &= \operatorname{tr}(\phi_{12}\phi_{23}\phi_{31}) \end{aligned}$$</td><td>Gauge group $\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(1)$
$\langle\phi_{12}\rangle=\begin{pmatrix}1 & 0\end{pmatrix}$
$\langle\phi_{23}\rangle=\begin{pmatrix}0\\1\end{pmatrix}$
$\langle\phi_{31}\rangle=\begin{pmatrix}0\end{pmatrix}$</td></tr>
<tr><td>Quiver</td><td>$$\begin{tikzcd} & ③ \arrow[dl, "\phi_{31}"'] & \\ ① \arrow[rr, "\phi_{12}"] & & ② \arrow[ul, "\phi_{23}"'] \end{tikzcd} \qquad \begin{aligned} \phi_{12}\phi_{23} &= 0 \\ \phi_{23}\phi_{31} &= 0 \\ \phi_{31}\phi_{12} &= 0 \end{aligned}$$</td><td>$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "(0)"'] & \\ \mathbb{C} \arrow[rr, "(1\ 0)"] & & \mathbb{C}^2 \arrow[ul, "\binom{0}{1}"'] \end{tikzcd}$$</td></tr>
</table>

Table 1: Examples for algebraic vs. representation data.

2. For each bifundamental chiral superfield draw an arrow. The field transforms in the antifundamental of the ith factor of the gauge group and the fundamental of the jth, and the arrow is directed from i to j.

3. For each F-term constraint $\partial W/\partial\phi_i$ define a relation for the corresponding arrows.

The resulting directed graph with relations (quiver) encodes all the information in (2.1).

A quiver representation is a choice of vector space for each node, and a choice of matrix for each arrow such that the product of the matrices satisfies the given relations. In other words it is precisely the representation data that has to complement the lagrangian (2.1).

The abstraction from representation theory data in SYM theories and quivers is illustrated in table 1. Two remarks are in order: First note that the relations of the quiver stand on the same footing as the directed graph. Drawing the graph and not writing down the relations is as nonsensical as leaving out random arrows or nodes. Second, I have ignored the D-term constraints. They are of course important, and are reflected by the D-flatness conditions on the SYM side and the choice of "stability" on the quiver side. However they will play no role in the following, so I will not review this here.

Now fix one SYM lagrangian resp. quiver, then it is obviously interesting to understand the set of possible representations. The all-important observation for the following is that there is an obvious notion of "map" from one representation to the other. Such a map is linear on the vector space attached to each node, and each square induced by an arrow has to commute. This is the straightforward generalization of the idea that one representation is a subrepresentation of another if it is contained as a block in the defining matrices. This notion of "map" is well-known in the representation theory of quivers, and turns the set[2] of representations into the category of representations. All that seems rather trivial, but being able to do "linear algebra" ultimately leads us to the derived category of such representations.

[2]As usual we assume that all categories are small.

2.2 The conjecture

We want to understand Seiberg duality, that is in the simplest case $\mathrm{U}(N_c)$ dual to $\mathrm{U}(N_f - N_c)$ + superpotential. So it connects two representations of different theories.

The new approach of [2] is to ask what remains on the purely algebraic level, without the representation theory data. Surely the dual lagrangians must be somehow connected, although their representation theory is very different, one is defined for all N_f, N_c and the other only for $N_f > N_c$.

In the language of quivers, a Seiberg dual pair Q_1, Q_2 has different representation categories: Q_1-rep $\neq$ Q_2-rep. It is not at all obvious what Q_1 and Q_2 should have in common.

We need some additional input. Consider type-IIB string theory compactified on some Calabi-Yau 3-fold, and add D-branes preserving half of the supersymmetry. Then one gets a $\mathcal{N} = 1$ low energy effective field theory, and we can try to study the field theories by analyzing the different D-brane configurations.

Now it was argued that the correct framework for studying the D-brane configurations is $\mathrm{D}^b(\mathrm{Coh}X)$, the bounded derived category of coherent sheaves on the Calabi-Yau manifold X. This would be a terribly esoteric approach were it not for the observation of [4] that (at least for some varieties X) the derived category $\mathrm{D}^b(\mathrm{Coh}X)$ is equivalent to $\mathrm{D}^b(Q\text{-rep})$ for some quiver Q. This quiver just has to be the one of the low energy effective field theory.

But the quiver Q is not uniquely determined, and in general there are different quivers Q_i having equivalent derived categories $\mathrm{D}^b(Q_i\text{-rep})$. Can they all define the same low-energy physics? Of course they should. Note that we no longer have to talk about coherent sheaves or Calabi-Yau manifolds, but can turn this into a pure gauge-theory statement:

Conjecture 1 (Berenstein-Douglas). *Two quivers Q_1, Q_2 are Seiberg dual if and only if* $\mathrm{D}^b(Q_1\text{-rep}) \simeq \mathrm{D}^b(Q_2\text{-rep})$.

Please note that there is a lot more information in this conjecture than a prescription like "flip some arrows and fix up the superpotential". There is no ambiguity, no guesswork involved. The appearance (or not) of superpotentials is completely forced on you.

This is very easy to state, but not at all obvious how those quivers look like in general. It is not even clear that what you would naively write down as BDS-dual quivers have equivalent derived representation categories. I will explore this in the following.

3. The Berenstein-Douglas toy quiver

The only example for a pair of BDS-dual quivers in [2] is the following:

$$\begin{aligned} Q_1 &\stackrel{\text{def}}{=} 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \\ Q_2 &\stackrel{\text{def}}{=} 1 \xleftarrow{\phi_{21}} 2 \xleftarrow{\phi_{32}} 3 \quad \phi_{32}\phi_{21} = 0\,. \end{aligned} \tag{3.1}$$

There is no oriented cycle, so one cannot write down a gauge invariant superpotential. But there is one relation in Q_2 which should derive from a superpotential. Obviously there is something missing, which I will explain in section 4.

For now it does serve as a very nice toy model where we can try to understand the equivalence of derived categories, see how the computations work and fix conventions. I will work purely in the derived category (which is more general and in some sense simpler), as opposed to the module theoretic approach of [2].

3.1 Path algebra, representations and modules

When writing down the superpotential constraints we always use an algebraic notation like $\langle \text{arrow}\,1 \rangle \langle \text{arrow}\,2 \rangle = 0$. Formalizing this leads to the following:

Definition 1. *The* path algebra $\mathbb{C}Q$ *of a quiver* Q *is the algebra (over* $\mathbb{C}$*) generated by the arrows of the quiver, and subject to the relations explained below. Here "arrows" stands for two kinds of arrows:*

- *The arrows* ϕ_{ij} *in the quiver, going from node* i *to node* j*.*
- *For each node* i *include a "zero length" arrow* e_i*, not drawn in the quiver diagram.*

The relations between the generators of the path algebra are the relations in the quiver. In addition to that we demand that every product is zero unless the arrows fit together, i.e. the only nonzero products are $\phi_{ij}\phi_{jk}$, $e_i\phi_{ij} \stackrel{\text{def}}{=} \phi_{ij}$, $\phi_{ij}e_j \stackrel{\text{def}}{=} \phi_{ij}$, $e_i^2 = e_i$.

Note that the composition of arrows is in the "intuitive" order, which is *opposite* to the composition of functions: If you have $f : X \to Y$ and $g : Y \to Z$ then usually their composition is denoted $g \circ f$. This means that in the representation of a quiver, the vectors attached to the nodes should be thought of as row vectors, and the matrices act by right-multiplication.

Example 1. *Take the quiver* $Q = Q_1$ *of eq.* (3.1)*. Then* $\mathbb{C}Q$ *is* 6*-dimensional as a vector space:*

$$\mathbb{C}Q = \text{span}_{\mathbb{C}}(e_1, e_2, e_3, \alpha, \beta, \alpha\beta) \tag{3.2}$$

What is it good for? Introducing the path algebra seems to be completely formal. Although not obvious, it does simplify the following computations considerably. The reason for this is the following elementary fact:

Theorem 1. *Let* Q *be an arbitrary quiver. Then the category of representations is the same as the category of* $\mathbb{C}Q$*-modules:*

$$Q\text{-rep} = \mathbb{C}Q\text{-mod} \tag{3.3}$$

So instead of dealing with representations we can work with modules. This is good because we have the following class of manageable modules:

Example 2. *Let* $P_i \stackrel{\text{def}}{=} (\mathbb{C}Q)e_i$*, the paths ending at node* i*. Then* P_i *is a* $\mathbb{C}Q$ *left module in the obvious sense:*

$$\phi \cdot (pe_i) = (\phi p)e_i \qquad \forall \phi, p \in \mathbb{C}Q \tag{3.4}$$

If you have a path from node i to j, then you can multiply P_i on the right and land in P_j. This is a $\mathbb{C}Q$ module homomorphism, and moreover all such homomorphisms come from such paths:

Theorem 2. $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}Q\text{-mod}}(P_i, P_j) = \#\big\{\text{Independent paths } i \to j\big\}$

Of course there are other $\mathbb{C}Q$-modules. But they are not important because of the following fact:

Theorem 3. *The P_i are projective $\mathbb{C}Q$-modules, and every projective module is a direct sum of P_i's.*

In the derived category $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$ every object is isomorphic to its projective resolution, so it suffices to know just the projective modules.

3.2 The derived category

So what is $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$, the derived category of $\mathbb{C}Q$-modules? Well to define the category you need the objects and the morphisms. The first is the easy part:

Definition 2. *The objects of* $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$ *are bounded complexes of $\mathbb{C}Q$-modules, i.e. a chain of modules and module homomorphisms*

$$\cdots \xrightarrow{d_{n-2}} M_{n-1} \xrightarrow{d_{n-1}} M_n \xrightarrow{d_n} M_{n+1} \xrightarrow{d_{n+1}} \cdots \tag{3.5}$$

such that going twice is zero, and only finitely many M_n are nonzero.

Now there is an obvious notion of "map" from one complex $f : M_\bullet \to N_\bullet$, given by maps of the modules such that

$$\begin{tikzcd} \cdots \arrow[r] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & M_n \arrow[r] \arrow[d, "f_n"'] & M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & \cdots \\ \cdots \arrow[r] & N_{n-1} \arrow[r] & N_n \arrow[r] & N_{n+1} \arrow[r] & \cdots \end{tikzcd} \tag{3.6}$$

commutes. Those chain maps are *not* the morphisms in the derived category. The derived category contains less information than the category of chain complexes. We get the morphisms of the derived category if we apply the following to the set of chain maps:

1. "Invert quasi-isomorphisms"
   Maybe there is no (nonzero) chain map $M_\bullet \to N_\bullet$, but there exists another complex $M'_\bullet$ with the same homology and a nonzero chain map $f' : M'_\bullet \to N_\bullet$. In the derived category you include f' in the morphisms from M to N.

   This sounds horribly complicated, but if all modules in the complex are already projective then this cannot happen. This is the reason for working only with the modules P_i.

2. "Go to the homotopy category of chain complexes"
   There is an equivalence relation (homotopy) on the chain maps that you have to mod out to get the morphisms of the derived category. A chain map $f : M_\bullet \to N_\bullet$ is homotopic to the zero map if there exists maps $s_n : M_n \to N_{n-1}$ such that

$$f_n = s_{n+1} d_n + d_{n-1} s_n \tag{3.7}$$

   In pictures:

$$\begin{tikzcd}
\cdots \arrow[r] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, dotted, "s_{n-1}"'] & M_n \arrow[r] \arrow[d, "f_n"] \arrow[dl, dotted, "s_n"'] & M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] \arrow[dl, dotted, "s_{n+1}"'] & \cdots \arrow[dl, dotted, "s_{n+2}"'] \\
\cdots \arrow[r] & N_{n-1} \arrow[r] & N_n \arrow[r] & N_{n+1} \arrow[r] & \cdots
\end{tikzcd} \tag{3.8}$$

3.3 Computations in the derived category

Enough of the theory, lets try to see how this works (we'll need the results of this section later). Fix $Q \stackrel{\text{def}}{=} Q_1$ of eq. (3.1), and consider the following 3 objects of $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$:

$$T_1 \stackrel{\text{def}}{=} 0 \longrightarrow 0 \longrightarrow \underline{P_1} \longrightarrow 0 \tag{3.9}$$

$$T_2 \stackrel{\text{def}}{=} 0 \longrightarrow 0 \longrightarrow \underline{P_3} \longrightarrow 0 \tag{3.10}$$

$$T_3 \stackrel{\text{def}}{=} 0 \longrightarrow P_2 \xrightarrow{\beta} \underline{P_3} \longrightarrow 0 \tag{3.11}$$

where the nonzero map in T_3 is right-multiplication by β, in abuse of notation again called β. The underlined entry denotes the one at position 0.

I want to identify the morphisms $\mathrm{Hom}(T_a[k], T_b) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{D}^b(\mathbb{C}Q\text{-mod})}(T_a[k], T_b)$ in the derived category ($T_a[k]$ is T_a shifted by k positions to the left). Here are some examples, the computation is always along these lines:

$\mathrm{Hom}(T_2, T_1) = 0$ because there is no module-homomorphism $P_2 \to P_1$ (there is no path in the quiver from node 2 to 1).

$\mathrm{Hom}(T_1, T_1) = \mathrm{span}(e_1) \simeq \mathbb{C}$ because the only module-homomorphism $P_1 \to P_1$ is multiplication with a constant. Or in other words, e_1 is the only path from node 1 to itself. Note that — again by abuse of notation — I also denote the morphism in the derived category by e_1.

$\mathrm{Hom}(T_1, T_2) = \mathrm{span}(\alpha\beta) \simeq \mathbb{C}$ by the same reason as above; The map cannot be homotopic to the zero map because too many entries are zero.

$\mathrm{Hom}(T_2, T_3) \simeq \mathbb{C}$ coming from multiplying P_3 by a constant. There is no homomorphism $P_3 \to P_2$ so there is no nontrivial chain homotopy.

$\mathrm{Hom}(T_1, T_3) = 0$ because the possible chain map is homotopic to zero:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "0"'] & \underline{P_1} \arrow[r] \arrow[d, "\alpha\beta"] \arrow[dl, dotted, "\alpha"'] & 0 \arrow[d] \arrow[dl, dotted, "0"'] \\
0 \arrow[r] & P_2 \arrow[r, "\beta"] & \underline{P_3} \arrow[r] & 0
\end{tikzcd} \tag{3.12}$$

$\mathrm{Hom}(T_1[1], T_3) = 0$ since — although there would be a nontrivial maps of the modules $\alpha : P_1 \to P_2$ — there are no chain maps:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lightning"] & \underline{0} \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & P_2 \arrow[r, "\beta"] & \underline{P_3} \arrow[r] & 0 \end{tikzcd} \qquad \alpha\beta \neq 0 \tag{3.13}$$

Along these lines one can determine all Hom's. The result is that

$$\mathrm{Hom}(T_a[i], T_b) = 0 \qquad \text{unless } i = 0 \tag{3.14}$$

so the only nontrivial morphisms are between the unshifted T's. Since the maximum width of the complexes is 2 every derived morphism is generated by a pair of module maps, which I denote $[f_1, f_2]$ in table 2.

A point worth noting is that although there are nonzero maps $T_1 \to T_2$ and $T_2 \to T_3$, the composition is zero in the derived category since $\mathrm{Hom}(T_1, T_3) = 0$. Although neither individual map is homotopic to zero, their composition is, see eq. (3.12).

$A \backslash B$	T_1	T_2	T_3
T_1	$[0, e_1]$	$[0, \alpha\beta]$	
T_2		$[0, e_3]$	$[0, e_3]$
T_3			$[e_2, e_3]$

Table 2: Summary of $\mathrm{Hom}(A, B)$ in $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$.

3.4 Fourier-Mukai transformation

Checking the equivalence of the derived categories $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ and $\mathrm{D}^b(\mathbb{C}Q_2\text{-mod})$ just from the definition certainly would be a formidable task. However there is some more machinery that makes this actually possible, Fourier-Mukai transformations. The single most useful thing about the whole derived category business is that Fourier-Mukai induces equivalences of derived categories (see e.g. [5]).

Now for the derived categories of quiver path algebra modules this boils down to something quite manageable: If you have an element $T \in \mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ (i.e. a complex of $\mathbb{C}Q_1$-modules) then you get an algebra $\mathrm{Hom}(T, T) \stackrel{\text{def}}{=} \mathrm{End}(T)$. If this algebra is isomorphic to $\mathbb{C}Q_2$ and if T has some nice properties then $\mathbb{C}Q_1$-mod and $\mathbb{C}Q_2$-mod are derived equivalent.

To be precise [6] showed the following:

Definition 3. $T \in \mathrm{D}^b(\mathbb{C}Q\text{-mod})$ *is called a* tilting complex *if*

1. $\mathrm{Hom}(T[i], T) = 0$ *for* $i \neq 0$
2. *Summands of direct sums of copies of* T *generate* $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$.

Theorem 4. *The derived categories* $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ *and* $\mathrm{D}^b(\mathbb{C}Q_2\text{-mod})$ *are equivalent if and only if there exists a tilting complex* $T \in \mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ *such that* $\mathrm{End}(T) \simeq \mathbb{C}Q_2$.

So in our case (the quivers in eq. (3.1)) set $T = T_1 \oplus T_2 \oplus T_3 \in \mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ with the T_i defined in eq. (3.11).

I claim that T is a tilting complex: There are no Hom's between $T[i]$ and T because of eq. (3.14), it remains to show that one can generate $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ from T_1, T_2 and T_3. For that it suffices to generate the stalk complexes for P_1, P_2 and P_3, i.e. complexes of the form $0 \to \underline{P_i} \to 0$ (by abuse of notation again denoted P_i).

The only nontrivial task is to generate the stalk complex of P_2. Here "generate" means to generate as derived category, so it includes operations like the shift and mapping cones. Let $f = [0, e_3]$ the generator of $\mathrm{Hom}(T_2, T_3)$, then by definition the mapping cone is

$$
M(f) = \qquad 0 \longrightarrow P_2 \oplus P_3 \xrightarrow{\left(\begin{smallmatrix} \beta \\ e_3 \end{smallmatrix}\right)} \underline{P_3} \longrightarrow 0 \tag{3.15}
$$

Now in the derived category $M(f) \simeq P_2[1]$. To see this check that the compositions of the obvious chain maps $M(f) \rightleftarrows P_2[1]$ are homotopic to the identity:

$$
\begin{tikzcd}
0 \arrow[r] & P_2 \oplus P_3 \arrow[r, "\left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & \underline{P_3} \arrow[r] \arrow[d] \arrow[ddl, dotted, "(0\ 1)"] & 0 \\
0 \arrow[r] & P_2 \arrow[r] \arrow[d, "(1\ -\beta)"'] & \underline{0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_2 \oplus P_3 \arrow[r, "\left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right)"] & \underline{P_3} \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & P_2 \arrow[r] \arrow[d, "(1\ -\beta)"'] & \underline{0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_2 \oplus P_3 \arrow[r, "\left(\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] & \underline{P_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_2 \arrow[r] & \underline{0} \arrow[r] & 0
\end{tikzcd}
\tag{3.16}
$$

Thus $T = T_1 \oplus T_2 \oplus T_3$ is a tilting complex. We determined $\mathrm{End}(T)$ in section 3.3, it is the path algebra of Q_2. The 3 orthogonal projectors correspond to the zero length arrows at the nodes, and the 2 remaining endomorphisms are just the arrows between the nodes (satisfying the relation that their composition is 0). With other words $\mathrm{End}(T) \simeq \mathbb{C}Q_2$ and therefore $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod}) \simeq \mathrm{D}^b(\mathbb{C}Q_2\text{-mod})$.

4. Physical quivers

4.1 Quivers with superpotential

It is not true that every quiver corresponds to an $\mathcal{N} = 1$ gauge theory, only the subset of quivers where the relations are derived from a superpotential do.

Definition 4. *A* superpotential W *(for some oriented graph* Γ*) is the trace over a linear combination of oriented cycles, i.e.*

$$
W \in \mathrm{tr} \bigoplus e_i \, (\mathbb{C}\Gamma) \, e_i \tag{4.1}
$$

Of course at the level of the path algebra, the trace is just a formal function with the cyclic permutation property.

Then a $\mathcal{N} = 1$ gauge theory corresponds to

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Definition 5. *A* quiver with superpotential W *is a quiver such that the relations are*[3] $\partial W / \partial \alpha_i = 0$, *where* α_i *are the arrows of the quiver.*

4.2 A sample duality pair

As already mentioned the BDS-dual pair in eq. (3.1) is not a pair of "quivers with superpotential". The obvious thing to do is to close up Q_2 with a third arrow. But this changes the derived category and $\mathrm{D}^b(Q_1\text{-rep}) \not\simeq \mathrm{D}^b(Q_2'\text{-rep})$, as I will demonstrate in section 6.1.

The simplest example for actual BDS-dual quivers with superpotential is the following:

$$
Q_1 \stackrel{\mathrm{def}}{=} \begin{tikzcd} ① \arrow[d, "\alpha"'] & \\ ② \arrow[d, "\beta"'] & ④ \arrow[l, "\delta"'] \\ ③ \arrow[uu, bend left=60, "\gamma"] & \end{tikzcd}
\qquad
\begin{aligned} \alpha\beta &= 0 \\ \beta\gamma &= 0 \\ \gamma\alpha &= 0 \end{aligned}
\qquad W_1 = \operatorname{tr}\left(\alpha\beta\gamma\right)
$$

$$
Q_2 \stackrel{\mathrm{def}}{=} \begin{tikzcd} ① & \\ ② \arrow[u, "\phi_{21}"] \arrow[r, "\phi_{24}"] & ④ \arrow[dl, "\phi_{43}"] \\ ③ \arrow[u, "\phi_{32}"] & \end{tikzcd}
\qquad
\begin{aligned} \phi_{32}\phi_{24} &= 0 \\ \phi_{24}\phi_{43} &= 0 \\ \phi_{43}\phi_{32} &= 0 \end{aligned}
\qquad W_2 = \operatorname{tr}\left(\phi_{32}\phi_{24}\phi_{43}\right) \tag{4.2}
$$

Note that the quivers can be distinguished by the direction of the arrow not in the cycle. The remainder of this section will be devoted to proving their derived equivalence, of course by using theorem 4.

The tilting complex. I claim that $T \stackrel{\mathrm{def}}{=} T_1 \oplus T_2 \oplus T_3 \oplus T_4 \in \mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$ is a tilting complex, where

$$
\begin{aligned}
T_1 &\stackrel{\mathrm{def}}{=} \quad 0 \longrightarrow \underline{P_1} \longrightarrow 0 \longrightarrow 0 \\
T_2 &\stackrel{\mathrm{def}}{=} \quad 0 \longrightarrow \underline{P_1 \oplus P_4} \xrightarrow{\left(\begin{smallmatrix}\alpha \\ \delta\end{smallmatrix}\right)} P_2 \longrightarrow 0 \\
T_3 &\stackrel{\mathrm{def}}{=} \quad 0 \longrightarrow \underline{P_3} \longrightarrow 0 \longrightarrow 0 \\
T_4 &\stackrel{\mathrm{def}}{=} \quad 0 \longrightarrow \underline{P_4} \longrightarrow 0 \longrightarrow 0
\end{aligned} \tag{4.3}
$$

First we need to show that $\mathrm{Hom}(T[k], T) = 0 \; \forall i \neq 0$. This is straightforward, I discuss the two prototypical sample cases:

[3]The authors conventions are that derivations act from the left

$\mathrm{Hom}(T_1[-1], T_2) = 0$ since the chain map is homotopic to zero:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \underline{0} \arrow[r] \arrow[d, "0"] \arrow[dl, dotted, "0"'] & P_1 \arrow[r] \arrow[d, "\alpha"] \arrow[dl, dotted, "(1\ 0)"'] & 0 \arrow[d] \arrow[dl, dotted, "0"'] \\ 0 \arrow[r] & \underline{P_1 \oplus P_4} \arrow[r, "\left(\begin{smallmatrix}\alpha\\ \delta\end{smallmatrix}\right)"'] & \underline{P_2} \arrow[r] & 0 \end{tikzcd} \tag{4.4}$$

$\mathrm{Hom}(T_2[1], T_3) = 0$ since there is no chain map:

$$\begin{tikzcd} 0 \arrow[r] & P_1 \oplus P_4 \arrow[r, "\left(\begin{smallmatrix}\alpha\\ \delta\end{smallmatrix}\right)"] \arrow[d] \arrow[dr, phantom, "\lightning"] & \underline{P_2} \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \underline{P_3} \arrow[r] & 0 \end{tikzcd} \qquad \left(\begin{smallmatrix}\alpha\\ \delta\end{smallmatrix}\right)\beta = \left(\begin{smallmatrix}0\\ \delta\beta\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right) \tag{4.5}$$

Second we need to check that we can generate the whole $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$. As in section 3.4 the only possible problem is to generate the stalk complex of P_2. But we can get this as the mapping cone of the obvious map $f : T_2 \to T_1 \oplus T_4$:

$$\begin{aligned} M(f) &= 0 \longrightarrow P_1 \oplus P_4 \xrightarrow{\left(\begin{smallmatrix}\alpha & 1 & 0\\ \delta & 0 & 1\end{smallmatrix}\right)} \underline{P_2 \oplus P_1 \oplus P_4} \longrightarrow 0 \\ &= 0 \longrightarrow \underline{P_2} \longrightarrow 0 \end{aligned} \tag{4.6}$$

The dual quiver. We have to determine $\mathrm{End}(T)$. The nonvanishing morphisms between the T_i are listed in table 3 using the same notation as in table 2. Let $\phi_{ab} \in \mathrm{Hom}(T_a, T_b)$ be the generator, then they satisfy the following relations:

$\phi_{31} = \phi_{32}\phi_{21}$ by matrix multiplication.

$\phi_{32}\phi_{24} = 0$ by matrix multiplication.

$\phi_{24}\phi_{43} = 0$ since the composition is in $\mathrm{Hom}(T_2, T_3) = 0$, the chain map is homotopic to zero.

$\phi_{43}\phi_{32} = 0$ since $\delta\beta\gamma = 0$.

After eliminating ϕ_{31} in the endomorphism algebra via the first relation we see that $\mathrm{End}(T) = \mathbb{C}Q_2$ of eq. (4.2).

$A\backslash B$	T_1	T_2	T_3	T_4
T_1	$[1,0]$			
T_2	$[\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right), 0]$	$[\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right), 1]$		$[\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right), 0]$
T_3	$[\gamma, 0]$	$[\left(\begin{smallmatrix}\gamma & 0\end{smallmatrix}\right), 0]$	$[1,0]$	
T_4			$[\delta\beta, 0]$	$[1,0]$

Table 3: Summary of $\mathrm{Hom}(A, B)$ in $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod})$.

4.3 Beyond tilting modules

In the summand T_2 of the tilting complex eq. (4.3) the map $\left(\begin{smallmatrix}\alpha \\ \delta\end{smallmatrix}\right) P_1 \oplus P_4 \to P_2$ is neither injective nor surjective, e.g. $e_2 \in P_2$ is not in the image and it maps $(\gamma, 0) \in P_1 \oplus P_4$ to zero. This means that the complex T_2 has non-zero homology at positions 0 and 1, so it is not the stalk complex of some tilting module. Although this does not prove that there might not be some other tilting module it illustrates that the derived category approach is more powerful than the one of [2].

In the case of toric dual quivers that I will treat next it will also be the case that the tilting complex has homology in more than one position.

However in all the examples that I will treat it suffices to use a tilting complex that consists of the stalk complexes of the projective modules, except one. Had we chosen to use only the P_i stalk complexes this would have also been a tilting complex, but the dual quiver would be the original one. In this sense we always dualize only a single node of the quiver, i.e. a single gauge group.

5. Toric duality is Seiberg duality

It has been argued that toric duality is Seiberg duality. This provides us with more examples of dual quivers, provided that we believe in the algorithm to read off the gauge theory data from the toric variety.

5.1 Review of toric duality

Toric duality was suggested in [7] as a new gauge theory duality coming from different resolutions of a $\mathbb{C}^3$-orbifold.

For concreteness consider the toric singularity $Z = \mathbb{C}^3/Z_3 \times \mathbb{Z}_3$ depicted in figure 1. Pick two different resolutions Z_1 and Z_2. Then a subvariety (given by a subfan) $X_1 \subset Z_1$ and $X_2 \subset Z_2$ are what I will call *weak toric dual*. By the inverse algorithm of [7] one can associate a gauge theory to X_1 and X_2, and those are also called weak toric dual. Since X_1 and X_2 do not really have anything in common, weak toric duality is not terribly interesting.

The interesting case is when X_1 and X_2 are both the same space (a line bundle over a toric surface), but embedded in different resolutions Z_1 and Z_2. Then the gauge theory obviously should be the same, while one in general finds different quivers. I will call this case *strong toric duality*, or just toric duality. Note that this is the case depicted in figure 1, the two bases of the line bundle are isomorphic toric varieties (as was noted in [8]).

In [9, 10] it was then argued that strong toric dual quivers are Seiberg dual gauge theories (slightly different dualities were considered in [11]). If you believe that the quiver associated to the toric variety is the correct one then this follows from the one-line argument that (see [2])

$$\mathrm{D}^b(\mathbb{C}Q_1\text{-mod}) \simeq \mathrm{D}^b(\mathrm{Coh}X_1) = \mathrm{D}^b(\mathrm{Coh}X_2) \simeq \mathrm{D}^b(\mathbb{C}Q_2\text{-mod}) \tag{5.1}$$

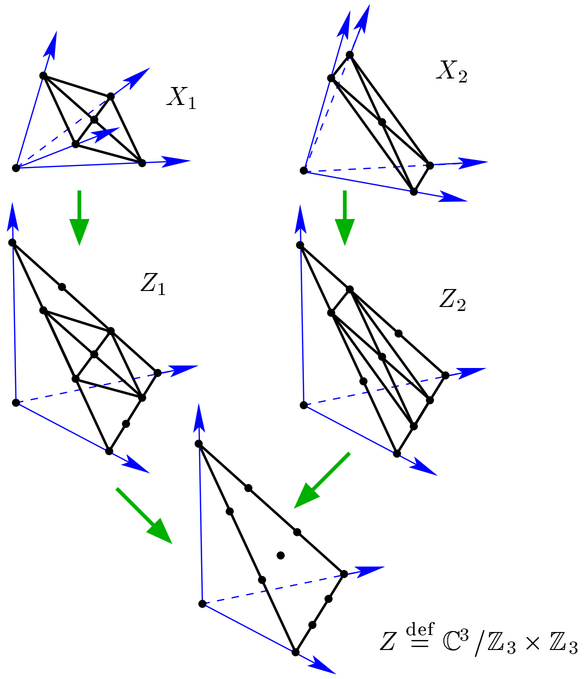
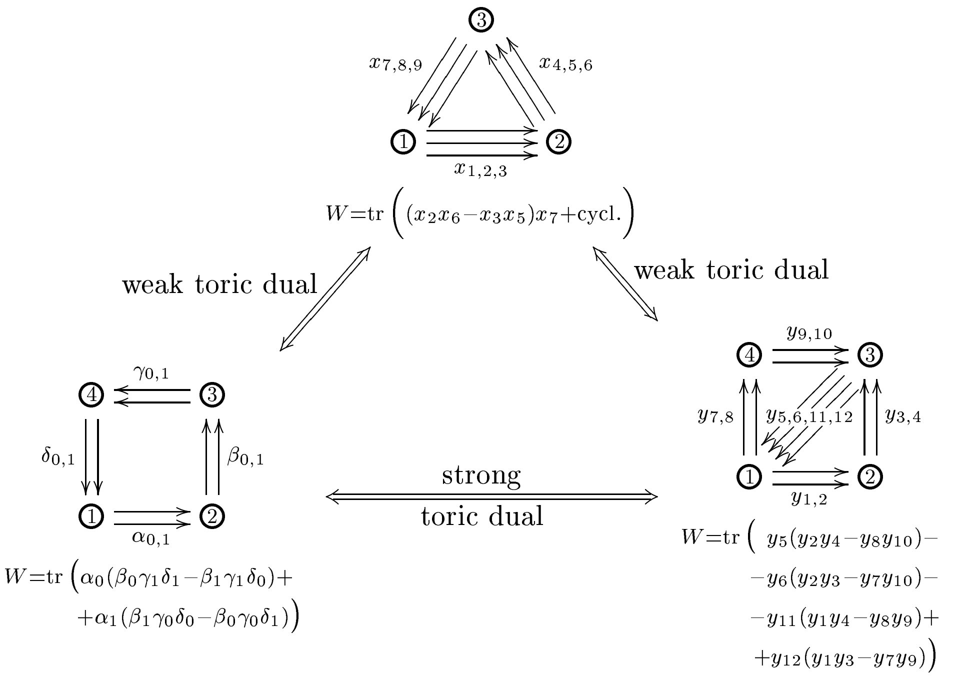


Figure 1: X_1 and X_2 are subvarieties of Z_1 and Z_2, which are both resolutions of Z.

By the algorithm of [7] (which I will not review here) one can for example find toric dualities between the following quivers:



(5.2)

In the following we will see (section 6.2) that the weakly dual quivers have non-equivalent derived categories, as expected. In the rest of this section I will demonstrate that the strong toric dual quivers above are BDS-dual.

5.2 Toric duality by tilting

It is a nice consistency check to actually show that the strong toric dual quivers of eq. (5.2) are BDS-dual, as I will do now. The starting point is the following quiver (the left one in eq. (5.2)) with the relations derived from the superpotential:

$$Q \overset{\text{def}}{=} \begin{tikzcd} ④ \arrow[d, shift right, "\delta_{0,1}"'] \arrow[d, shift left] & ③ \arrow[l, shift right, "\gamma_{0,1}"'] \arrow[l, shift left] \\ ① \arrow[r, shift left] \arrow[r, shift right, "\alpha_{0,1}"'] & ② \arrow[u, shift left] \arrow[u, shift right, "\beta_{0,1}"'] \end{tikzcd} \qquad \begin{array}{ll} \beta_0\gamma_1\delta_1 = \beta_1\gamma_1\delta_0 & \beta_1\gamma_0\delta_0 = \beta_0\gamma_0\delta_1 \\ \gamma_0\delta_1\alpha_1 = \gamma_1\delta_1\alpha_0 & \gamma_1\delta_0\alpha_0 = \gamma_0\delta_0\alpha_1 \\ \delta_0\alpha_1\beta_1 = \delta_1\alpha_1\beta_0 & \delta_1\alpha_0\beta_0 = \delta_0\alpha_0\beta_1 \\ \alpha_0\beta_1\gamma_1 = \alpha_1\beta_1\gamma_0 & \alpha_1\beta_0\gamma_0 = \alpha_0\beta_0\gamma_1 \end{array} \tag{5.3}$$

Technically the path algebra is much harder to analyze since there are arbitrary long paths, i.e. $\mathbb{C}Q$ is an infinite-dimensional algebra.

Since we cannot simply enumerate all paths I will first describe some of the structure of the path algebra. First note that the relations allow us to sort the indices at odd and at even positions (e.g. counting the individual arrows from left to right starting at 1), not more and not less. This implies the

Definition 6 (Normal form for paths in $\mathbb{C}Q$). *Consider the relation $\succ$ that compares the indices of the arrows of the quiver:*

$$v \succ w \overset{def}{\iff} v \in \{\alpha_1, \beta_1, \gamma_1, \delta_1\} \quad \textit{and} \quad w \in \{\alpha_0, \beta_0, \gamma_0, \delta_0\} \tag{5.4}$$

and let $\preceq$ be the complement (comparing indices via $\leq$). Then for every path

$$p = p_1 p_2 \cdots p_k \in \mathbb{C}Q \tag{5.5}$$

there exists a unique representative of the form $\tilde{p} = \tilde{p}_1 \tilde{p}_2 \cdots \tilde{p}_k$ such that the even and odd $\tilde{p}$'s are $\preceq$-ordered:

$$\tilde{p}_{2i} \preceq \tilde{p}_{2i+2} \quad \textit{and} \quad \tilde{p}_{2i+1} \preceq \tilde{p}_{2i+3} \tag{5.6}$$

The normal form implies another useful observation:

Lemma 1. *The path algebra $\mathbb{C}Q$ is $\mathbb{Z} \oplus \mathbb{Z}$-graded via*

$$grade(p) = (\textit{length of } p,\ \textit{sum of the indices of the individual arrows}) \tag{5.7}$$

This is useful because we can sort the paths, for example first sort by length and then break ties by comparing the sum of the indices. Then we can understand the following:

Lemma 2.

$$(\delta_0 - \delta_1)x \neq 0 \quad \forall x \in e_1(\mathbb{C}Q) - \{0\} \tag{5.8}$$

Proof. To show that a given sum-of-paths is not zero we just have to show that the "smallest" path is nonzero, i.e. it suffices to show that $\delta_0 x \neq 0$. But prefixing each path with δ_0 acts injectively on the normal forms. ∎

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The tilting complex. Again I will use theorem 4 to find a BDS-dual quiver. I claim that $T \overset{\text{def}}{=} T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is a tilting complex, where

$$
\begin{aligned}
T_1 &\overset{\text{def}}{=} & 0 \longrightarrow \underline{P_4 \oplus P_4} &\xrightarrow{\left(\begin{smallmatrix}\delta_0\\ -\delta_1\end{smallmatrix}\right)} P_1 \longrightarrow 0 \\
T_2 &\overset{\text{def}}{=} & 0 \longrightarrow \underline{P_2} &\longrightarrow 0 \longrightarrow 0 \\
T_3 &\overset{\text{def}}{=} & 0 \longrightarrow \underline{P_3} &\longrightarrow 0 \longrightarrow 0 \\
T_4 &\overset{\text{def}}{=} & 0 \longrightarrow \underline{P_4} &\longrightarrow 0 \longrightarrow 0
\end{aligned}
\tag{5.9}
$$

Again T_1 has homology at position -1 and 0, so again this tilting complex cannot simply be rephrased as a tilting module. Now first I have to show that the Hom's between shifted T_a's vanish. The nontrivial cases are

$\mathrm{Hom}(T_2[-1], T_1) = \mathrm{Hom}(T_3[-1], T_1) = \mathrm{Hom}(T_4[-1], T_1) = 0$ because possible chain maps are homotopic to zero.

$\mathrm{Hom}(T_1[1], T_2) = \mathrm{Hom}(T_1[1], T_3) = \mathrm{Hom}(T_1[1], T_4) = 0$ because there are no chain maps, for example take

$$
\begin{tikzcd}
0 \arrow[r] & P_4 \oplus P_4 \arrow[r, "\left(\begin{smallmatrix}\delta_0\\ -\delta_1\end{smallmatrix}\right)"] \arrow[d] & \underline{P_1} \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \underline{P_2} \arrow[r] & 0
\end{tikzcd}
\tag{5.10}
$$

then by lemma 2 commutativity requires $x = 0$.

$\mathrm{Hom}(T_1[1], T_1) = 0$ again since there is no chain map:

$$
\begin{tikzcd}
0 \arrow[r] & P_4 \oplus P_4 \arrow[r, "\left(\begin{smallmatrix}\delta_0\\ -\delta_1\end{smallmatrix}\right)"] \arrow[d] & \underline{P_1} \arrow[r] \arrow[d, "x"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \underline{P_4 \oplus P_4} \arrow[r, "\left(\begin{smallmatrix}\delta_0\\ -\delta_1\end{smallmatrix}\right)"] & P_1 \arrow[r] & 0
\end{tikzcd}
\tag{5.11}
$$

While the right square can be made commutative with $x \neq 0$, the left square can not (again by lemma 2).

Second I have to show that $T_1, \ldots, T_4$ generate the whole $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$. Again this is so because we can generate P_1 by the mapping cone of $T_1 \to T_4 \oplus T_4$.

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$A\backslash B$	T_1	T_2	T_3	T_4
T_1	$[\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),1]$			$y_9\stackrel{\text{def}}{=}[\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),0]$ $y_{10}\stackrel{\text{def}}{=}[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),0]$
T_2	$y_7\stackrel{\text{def}}{=}[\left(\begin{smallmatrix}\beta_1\gamma_0&\beta_0\gamma_0\end{smallmatrix}\right),0]$ $y_8\stackrel{\text{def}}{=}[\left(\begin{smallmatrix}\beta_1\gamma_1&\beta_0\gamma_1\end{smallmatrix}\right),0]$	$[1,0]$	$y_1\stackrel{\text{def}}{=}[\beta_0,0]$ $y_2\stackrel{\text{def}}{=}[\beta_1,0]$	
T_3			$[1,0]$	$y_3\stackrel{\text{def}}{=}[\gamma_0,0]$ $y_4\stackrel{\text{def}}{=}[\gamma_1,0]$
T_4		$y_5\stackrel{\text{def}}{=}[\delta_0\alpha_0,0]$ $y_6\stackrel{\text{def}}{=}[\delta_0\alpha_1,0]$ $y_{11}\stackrel{\text{def}}{=}[\delta_1\alpha_0,0]$ $y_{12}\stackrel{\text{def}}{=}[\delta_1\alpha_1,0]$		$[1,0]$

Table 4: Summary of generators of $\mathrm{Hom}(A,B)$ in $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$.

The dual quiver. Determining the dual quiver is again technically more complicated because there are infinitely many maps from each T_a to each T_b. So writing down infinitely many maps, almost all of which can be eliminated by relations, is obviously a bad idea. Instead we must be careful to identify the elements of $\mathrm{End}(T)$ which do not factor through other endomorphisms.

First consider T_2, T_3 and T_4. Since they are just stalk complexes the homomorphisms are generated by the paths in Q. They are summarized in table 4, where again I will use the notation $[x_1, x_2]$ (as in table 2) to denote the two nontrivial chain maps between the summands of T. More difficult are the Hom's involving T_1. I discuss the cases shown in table 4:

$\mathrm{Hom}(T_1,T_4)$: They are obviously generated by the projection on the first or second P_4 summand.

$\mathrm{Hom}(T_1,T_2)$ and $\mathrm{Hom}(T_1,T_3)$: Every such morphism factors through $\mathrm{Hom}(T_1,T_4)$, so I do not include them into the list of generators.

$\mathrm{Hom}(T_4,T_1)$: The simplest guess $[\left(\begin{smallmatrix}1&0\end{smallmatrix}\right),0]$ does not work since it is not a chain map. One has to "go around the square" once, so one can use a relation in the path algebra to generate a chain map, like

$$\begin{tikzcd} 0 \arrow[r] & \underline{P_4} \arrow[r] \arrow[d, "\left(\begin{smallmatrix}\delta_0\alpha_0\beta_1\gamma_0 & \delta_0\alpha_0\beta_0\gamma_0\end{smallmatrix}\right)"'] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{P_4 \oplus P_4} \arrow[r, "\left(\begin{smallmatrix}\delta_0\\-\delta_1\end{smallmatrix}\right)"] & P_1 \arrow[r] & 0 \end{tikzcd} \tag{5.12}$$

$$\left(\begin{smallmatrix}\delta_0\alpha_0\beta_1\gamma_0 & \delta_0\alpha_0\beta_0\gamma_0\end{smallmatrix}\right)\begin{pmatrix}\delta_0\\-\delta_1\end{pmatrix} = \delta_0\alpha_0(\beta_1\gamma_0\delta_0 - \beta_0\gamma_0\delta_1) = 0$$

However such a morphism factors through the $\mathrm{Hom}(T_4, T_2)$ to be discussed below, and is therefore not included in the table 4 of generators.

$\mathrm{Hom}(T_2, T_1)$: The possible chain maps

$$\begin{tikzcd} 0 \arrow[r] & \underline{P_2} \arrow[r] \arrow[d, "\begin{pmatrix} x_1 & x_2 \end{pmatrix}"'] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{P_4 \oplus P_4} \arrow[r, "\begin{pmatrix} \delta_0 \\ -\delta_1 \end{pmatrix}"] & P_1 \arrow[r] & 0 \end{tikzcd} \tag{5.13}$$

have to satisfy

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \delta_0 \\ -\delta_1 \end{pmatrix} = 0 \tag{5.14}$$

The minimal (path length 2) solutions are

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \in \mathrm{span}_{\mathbb{C}} \left(\begin{pmatrix} \beta_1\gamma_0 & \beta_0\gamma_0 \end{pmatrix}, \begin{pmatrix} \beta_1\gamma_1 & \beta_0\gamma_1 \end{pmatrix} \right) \tag{5.15}$$

and all longer (path length 6, 10, ...) solutions factor through those.

$\mathrm{Hom}(T_1, T_1)$ and $\mathrm{Hom}(T_3, T_1)$: These do not yield new generators for the same reason as $\mathrm{Hom}(T_4, T_1)$.

The relations in the dual quiver. I fixed a nice set of generators in table 4, but they are not independent. Using the relations in Q it is easy to check the following 10 relations in $\mathrm{End}(T)$:

$$\begin{array}{lllll} y_2y_4 = y_8y_{10} & y_1y_4 = y_8y_9 & y_4y_{11} = y_3y_{12} & y_5y_8 = y_6y_7 & y_6y_2 = y_{12}y_1 \\ y_2y_3 = y_7y_{10} & y_1y_3 = y_7y_9 & y_{11}y_8 = y_{12}y_7 & y_4y_5 = y_3y_6 & y_5y_2 = y_{11}y_1 \end{array} \tag{5.16}$$

The above relations hold already on the level of chain maps. But in the derived category we also have to identify homotopic maps, leading to additional relations in $\mathrm{End}(T)$. The two additional relations are

$$y_{10}y_6 = y_9y_{12} \qquad y_{10}y_5 = y_9y_{11} \tag{5.17}$$

For example to show that $y_{10}y_6 = y_9y_{12}$ one has to show that the difference of the chain maps is homotopic to zero. The homotopy is straightforward to find:

$$y_{10}y_6 - y_9y_{12} \sim 0 : \quad \begin{tikzcd} 0 \arrow[r] & \underline{P_4 \oplus P_4} \arrow[r, "\begin{pmatrix} \delta_0 \\ -\delta_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} \delta_0\alpha_1 \\ -\delta_1\alpha_1 \end{pmatrix}"'] & P_1 \arrow[r] \arrow[d] \arrow[dl, dotted, "\alpha_1"] & 0 \\ 0 \arrow[r] & \underline{P_2} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{5.18}$$

Together the relations eq. (5.16) and (5.17) are precisely the F-terms for the superpotential in the toric dual quiver in eq. (5.2). So in this case strong toric duality is a BDS-duality, and the equivalence of derived categories is induced by the tilting complex T above.

6. Non-dualities and invariants of the derived category

Clearly it is desirable to have simple criteria to check whether the derived categories can at all be derived equivalent.

6.1 Global dimension

As the simplest example of a conserved quantity I would like to consider the (finiteness of the) global dimension of the quiver. Take all possible modules of the path algebra, then for each one there is a minimal projective resolution. The maximal length is the global dimension of the quiver path algebra.

This is useful because of the following (see [12]):

Theorem 5. *Let $\mathbb{C}Q_1$ and $\mathbb{C}Q_2$ be finite dimensional algebras. If $\mathrm{D}^b(\mathbb{C}Q_1\text{-mod}) \simeq \mathrm{D}^b(\mathbb{C}Q_2\text{-mod})$ and $\mathbb{C}Q_1$ has finite global dimension then $\mathbb{C}Q_2$ also has finite global dimension.*

With the help of this theorem we can immediately show that "closing up the loop" in Q_2 of eq. (3.1) does change the derived category: let

$$Q_2 \overset{\text{def}}{=} \begin{array}{c} \begin{tikzcd} ① & ② \arrow[l, "\phi_{21}"'] & ③ \arrow[l, "\phi_{32}"'] \end{tikzcd} \\ \phi_{32}\phi_{21} = 0 \end{array} \qquad Q_3 \overset{\text{def}}{=} \begin{tikzcd} & ③ \arrow[dl, "\phi_{31}"'] & \\ ① \arrow[rr, "\phi_{12}"] & & ② \arrow[ul, "\phi_{23}"'] \end{tikzcd} \quad \begin{array}{l} \phi_{12}\phi_{23} = 0 \\ \phi_{23}\phi_{31} = 0 \\ \phi_{31}\phi_{12} = 0 \end{array} \tag{6.1}$$

and note that $\operatorname{gldim}\mathbb{C}Q_2 = 2$, $\operatorname{gldim}\mathbb{C}Q_3 = \infty$. Therefore $\mathrm{D}^b(\mathbb{C}Q_2\text{-mod}) \not\simeq \mathrm{D}^b(\mathbb{C}Q_3\text{-mod})$.

Note that it is well-known that the global dimension itself (if finite) is not preserved under equivalence of derived categories, only whether it is finite or infinite. It would be nice if one had a physical argument why this should be invariant.

6.2 K-theory

Let us for the moment go back to the string theoretic motivation and think of the $\mathcal{N} = 1$ SYM as the low energy effective action of D-branes on a Calabi-Yau manifold. Now consider two possible brane setups: In one case you allow only for (arbitrary numbers of) branes wrapping a single fixed cycle, and in the other case you allow branes wrapping all possible cycles. Will the two $\mathcal{N} = 1$ theories be dual? Of course they should not, the second case should contain a lot more physical information.

Put differently, in the first case I considered only multiples of one particular D-brane charge, while in the second I allowed all possible charges. Two D-brane categories should only be dual if their K-theory lattice is the same.

This fits very nicely with the mathematics of derived categories (see [12]). To each derived category $\mathrm{D}^b(A)$ we can associate the Grothendieck group $K_0(A)$ by modding out

1. Isomorphism

2. Elements of the form $[X] - [Y] + [Z]$ for each triangle $X \to Y \to Z \to X[1]$.

and this is the same as the K-group of A itself. So the derived category contains all the information of the K-groups.

Now again we can forget about the D-branes and work only with the derived quiver representations. The K-groups of $\mathrm{D}^b(\mathbb{C}Q\text{-mod})$ is the same as the K-theory of the path algebra. The K-theory of the algebra is generated by the projective modules, which have in our case a quite simple structure: They are in one-to-one correspondence with the nodes of the quiver, theorem 3. Thus

$$K_0(\mathbb{C}Q) = \mathbb{Z}^{\#\text{of nodes of } Q} \tag{6.2}$$

So we see immediately that the weakly toric dual quivers of eq. (5.2) cannot have equivalent derived categories since the number of nodes is different.

Also note that $K_0(\mathbb{C}Q)$ is always free, i.e. there is no torsion ($\mathbb{Z}_n$) subgroup. One should not expect that one can always understand the derived category of coherent sheaves by simply studying quivers.

Conclusions

I have reviewed the conjecture of Berenstein and Douglas [2] which can be seen as a precise and unambiguous definition of Seiberg duality. Despite the fearsome mathematical language involved it is impressive that it can be stated in one single line.

To actually apply the definition one has to unravel the words, which can lead do quite intricate computations. It is already nontrivial to check that it gives a reasonable answer for the original toy quiver of [2].

But to actually check the proposal one needs examples for dual gauge theories, that is quivers with superpotential. I gave a fairly simple example of such a BDS-dual quiver pair, and explicitly checked the equivalence of derived categories. For this we had to work really in the derived category and go beyond tilting modules.

Although it sounds scary it is actually quite feasible to show the equivalences of derived categories. To demonstrate the power of this approach I then tackled a technically much more complicated problem: I showed explicitly that a pair of toric dual quivers is BDS-dual. While the equivalence of derived categories was expected on general grounds it is a very nice check that the algorithm to associate the quiver to the toric variety is correct.

Finally I mentioned some invariants under derived equivalence, which are very useful if one wants to show that two quivers are *not* BDS-dual.

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